一组不等式的证明

徐一博 指导老师 余水能

(武汉二中,湖北 武汉 430010)

文[1]介绍了这样一组不等式:

已知:$x_i\in\mathbf{R}^+,i=1,2,\cdots,n,k\in\mathbf{N}^*,\sum\limits_{i=1}^{n}x_i=1$,则

1) $\prod\limits_{i=1}^{n}(\frac{1}{x_i}+x_i)\geqslant(n+\frac{1}{n})^n$;

2) $\prod\limits_{i=1}^{n}(\frac{1}{x_i^k}+1)\geqslant(n^k+1)^n$;

3) $\prod\limits_{i=1}^{n}(\frac{1}{x_i^k}+x_i^k)\geqslant(n^k+\frac{1}{n^k})^n$;

4) $\prod\limits_{i=1}^{n}(\frac{1}{x_i^k}-1)\geqslant(n^k-1)^n$;

5) $\prod\limits_{i=1}^{n}(\frac{1}{x_i}-x_i)\geqslant(n-\frac{1}{n})^n$(此处 $n\geqslant3$).

文[1]对5)给出了详细的初等证明,而对1)~4)未给出详尽的证明.本文则对前四个不等式进行证明.我们首先来证明两个引理.

引理1 $a_i,b_i\in\mathbf{R}^+,i=1,2,\cdots,n$,则$\prod\limits_{i=1}^{n}(a_i+b_i)^{\frac{1}{n}}\geqslant(\prod\limits_{i=1}^{n}a_i)^{\frac{1}{n}}+(\prod\limits_{i=1}^{n}b_i)^{\frac{1}{n}}$.

证明 此不等式等价于

$$\prod_{i=1}^{n}\left(\frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}}+\prod_{i=1}^{n}(\frac{b_i}{a_i+b_i})^{\frac{1}{n}}\leqslant1.$$

$$\because\prod_{i=1}^{n}\left(\frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}}\leqslant\frac{1}{n}\sum_{i=1}^{n}\frac{a_i}{a_i+b_i},$$

$\prod\limits_{i=1}^{n}(\frac{b_i}{a_i+b_i})^{\frac{1}{n}}\leqslant\sum\limits_{i=1}^{n}\frac{b_i}{a_i+b_i}$,(均值不等式)

$\therefore\prod\limits_{i=1}^{n}(\frac{a_i}{a_i+b_i})^{\frac{1}{n}}+\prod\limits_{i=1}^{n}(\frac{b_i}{a_i+b_i})^{\frac{1}{n}}\leqslant\frac{1}{n}(\sum\limits_{i=1}^{n}\frac{a_i}{a_i+b_i}+\sum\limits_{i=1}^{n}\frac{b_i}{a_i+b_i})=\frac{1}{n}\sum\limits_{i=1}^{n}\frac{a_i+b_i}{a_i+b_i}=\frac{1}{n}\cdot n=1$,引理1得证.

将引理1推广,又可得到下面引理2.

引理2 $a_{ij}\in\mathbf{R}^+,i=1,2,\cdots,n,j=1,2,\cdots,m$,则

$$\prod_{i=1}^{n}(\sum_{j=1}^{m}a_{ij})^{\frac{1}{n}}\geqslant\sum_{j=1}^{m}(\prod_{i=1}^{n}a_{ij})^{\frac{1}{n}}.$$

证明 此不等式等价于$\sum\limits_{j=1}^{m}\dfrac{(\prod\limits_{i=1}^{n}a_{ij})^{\frac{1}{n}}}{\prod\limits_{i=1}^{n}(\sum\limits_{j=1}^{m}a_{ij})^{\frac{1}{n}}}\leqslant1$.

$$\because\frac{(\prod\limits_{i=1}^{n}a_{ij})^{\frac{1}{n}}}{\prod\limits_{i=1}^{n}(\sum\limits_{j=1}^{m}a_{ij})^{\frac{1}{n}}}=\prod_{i=1}^{n}(\frac{a_{ij}}{\sum\limits_{j=1}^{m}a_{ij}})^{\frac{1}{n}}\leqslant\frac{1}{n}(\sum_{i=1}^{n}\frac{a_{ij}}{\sum\limits_{j=1}^{m}a_{ij}}),$$

$$\therefore\sum_{j=1}^{m}\frac{(\prod\limits_{i=1}^{n}a_{ij})^{\frac{1}{n}}}{\prod\limits_{i=1}^{n}(\sum\limits_{j=1}^{m}aij)^{\frac{1}{n}}}\leqslant\sum_{j=1}^{m}\left[\frac{1}{n}\left(\frac{a_{ij}}{\sum\limits_{j=1}^{m}a_{ij}}\right)\right]$$

$$=\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_{ij}}{\sum\limits_{j=1}^{m}a_{ij}}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{a_{ij}}{\sum\limits_{j=1}^{m}a_{ij}}=\frac{1}{n}\sum_{i=1}^{n}1=\frac{1}{n}\cdot n=1.$$

引理2得证.

在下面的证明中,要用到下面的一个结论 p:

令 $f(x)=x+\frac{1}{x},x\in(0,+\infty)$,对 $f(x)$取导,$f'(x)=1-x^{-2}$.

当 $x\in(0,1)$时,$f'(x)=1-\frac{1}{x^2}<0$,$f(x)$单调减;

当 $x\in(1,+\infty)$时,$f'(x)=1-\frac{1}{x^2}>0$,$f(x)$单调增;

当 $x=1$时,$f'(x)=0$,$f(x)$在 $x=1$时取最小值2.

下面对1)~4)给出证明

1)由引理1,$\prod\limits_{i=1}^{n}\left(x_1+\frac{1}{x_i}\right)\geqslant[(\prod\limits_{i=1}^{n}x_i)^{\frac{1}{n}}+\left(\prod\limits_{i=1}^{n}\frac{1}{x_i}\right)^{\frac{1}{n}}]^n$.

$$\because(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}\leqslant\frac{1}{n}\sum_{i=1}^{n}x_i=\frac{1}{n}\leqslant1.$$

由上结论,$f(x)$在(0,1)上单调减,故$f[(\prod\limits_{i=1}^{n}x_i)^{\frac{1}{n}}]\geqslant f\left(\frac{1}{n}\right)$,即

$$(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}+\frac{1}{(\prod\limits_{i=1}^{n}x_n)^{\frac{1}{n}}}\geqslant\frac{1}{n}+n,\text{故}$$

$$[(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}+\frac{1}{(\prod\limits_{i=1}^{n}x_i)^{\frac{1}{n}}}]^n\geqslant\left(\frac{1}{n}+n\right)^n.$$

2)同样,运用引理1,有

$$\prod_{i=1}^{n}\left(\frac{1}{x_i^k}+1\right)\geqslant\left[\left(\prod_{i=1}^{n}\frac{1}{x_i^k}\right)^{\frac{1}{n}}+1\right]^n$$

$$=\left(\left[\frac{1}{(\prod\limits_{i=1}^{n}x_i)^{\frac{1}{n}}}\right]^k+1\right)^n$$

$$\geqslant\left(\left[\frac{1}{\frac{1}{n}\sum\limits_{i=1}^{n}x_i}\right]^k+1\right)^n=(n^k+1)^n.$$

3)由引理1, $\prod_{i=1}^{n}\left(\frac{1}{x_i^k}+x_i^k\right)\geqslant\left[\left(\prod_{i=1}^{n}\frac{1}{x_i^k}\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}x_i^k\right)^{\frac{1}{n}}\right]^n$

$=\{f[(\prod_{i=1}^{n}x_i^k)^{\frac{1}{n}}]\}^n$.

$\because[(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}]^k\leqslant[\frac{1}{n}\sum_{i=1}^{n}x_i]^k=\frac{1}{n^k}\leqslant 1$,

$\therefore f[(\prod_{i=1}^{n}x_i^k)^{\frac{1}{n}}]=f\{[(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}]^k\}\geqslant f\left(\frac{1}{n^k}\right)$,

$\{f[(\prod_{i=1}^{n}x_i^k)^{\frac{1}{n}}]\}^n\geqslant[f(\frac{1}{n^k})]^n$,

即 $\prod_{i=1}^{n}(x_i^k+\frac{1}{x_i^k})\geqslant[f(\frac{1}{n^k})]^n=\left(n^k+\frac{1}{n^k}\right)^n$.

4)分二步证明.

①$k=1$时,原不等式等价于

$$\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right)\geqslant(n-1)^n.$$

由于 $\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right)=\prod_{i=1}^{n}\left(\frac{\sum_{j=1}^{n}x_j-x_i}{x_i}\right)$

$=\prod_{i=1}^{n}\frac{\sum_{i\neq j,j=1}^{n}x_i}{x_i}$

$\geqslant\prod_{i=1}^{n}\frac{(n-1)(\prod_{j\neq i,j=1}^{n}x_j)^{\frac{1}{n-1}}}{x_i}$

$=(n-1)^n\cdot\frac{\prod_{i=1}^{n}\left(\frac{\prod_{j=1}^{n}x_j}{x_i}\right)^{\frac{1}{n-1}}}{\prod_{i=1}^{n}x_i}$

$=(n-1)^n\left(\frac{(\prod_{j=1}^{n}x_j)^n}{\prod_{i=1}^{n}x_i}\right)^{\frac{1}{n}}/\prod_{i=1}^{n}x_i=(n-1)^n$,

即$k=1$时4)成立

②当$k>1$即$k\geqslant 2$时,由于

$(a^k-1)=(a-1)(\sum_{i=0}^{k-1}a^i)$,

那么, $\prod_{i=1}^{n}\left(\frac{1}{x_i^k}-1\right)=\prod_{i=1}^{n}\left[\left(\frac{1}{x_i}-1\right)\left(\sum_{j=0}^{k-1}(\frac{1}{x_i})^j\right)\right]$

$\geqslant(n-1)^n\cdot\prod_{i=1}^{n}[\sum_{j=0}^{k-1}(\frac{1}{x_i})^j]$(由①)

$\geqslant(n-1)^n\cdot\{\sum_{j=0}^{k-1}\left[\prod_{i=1}^{n}(\frac{1}{x_i})^j\right]^{\frac{1}{n}}\}^n$(由引理2)

$=(n-1)^n\cdot\left[\sum_{j=0}^{k-1}\left(\frac{1}{(\prod_{i=1}^{n}x_i)^{\frac{1}{n}}}\right)^j\right]^n$

$\geqslant(n-1)^n\cdot\left[\sum_{j=0}^{k-1}\left(\frac{1}{\frac{1}{n}\sum_{i=1}^{n}x_i}\right)^j\right]^n$

$=(n-1)^n(\sum_{j=0}^{k-1}n^j)^n=[(n-1)(\sum_{j=0}^{k-1}n^j)]^n$

$=(n^k-1)^n$,

即$k\geqslant 2$时4)成立

综上则有4)成立.

以上我们利用了引理1、引理2、均值不等式进行了整体上的调控,完成了对1)~4)的证明.另外,我们还可以对此不等式的条件进行探索性推广:

若$\sum_{i=1}^{n}x_i\leqslant 1,x_i\in R^+$,1)~5)仍然成立.

我们只需对$\sum_{i=1}^{n}x_i<1$给出证明.

事实上,$\sum_{i=1}^{n}x_i=\sum_{i=1}^{n-1}x_i+x_n<1$,可选取$x_n'$,$1>x_n'>x_n'$ (1)

使$\sum_{i=1}^{n-1}x_i+x_n'=1$,

显然,$1>x_n'^k>x_n^k$ (2)

对于1),由(1)与上述结论p,知$f(x_n')<f(x_n)$,即$x_n+\frac{1}{x_n}>\frac{1}{x_n'}+x_n'$,

$\therefore\left(\frac{1}{x_n}+x_n\right)\prod_{i=1}^{n-1}\left(\frac{1}{x_i}+x_i\right)\geqslant\left(\frac{1}{x_n'}+x_n'\right)\cdot\prod_{i=1}^{n-1}\left(x_i+\frac{1}{x_i}\right)\geqslant\left(n+\frac{1}{n}\right)^n$,

即1)成立.

即2),4),由(2)知,$\frac{1}{x_n^k}\pm 1>\frac{1}{(x_n')^k}\pm 1$,

$\therefore\prod_{i=1}^{n}\left(\frac{1}{x_i^k}\pm 1\right)>\left[\frac{1}{(x_n')^k}\pm 1\right]\prod_{i=1}^{n-1}\left(\frac{1}{x_i^k}\pm 1\right)\geqslant(n^k\pm 1)^n$ 成立.

对3),由(2)与结论p知$f[(x_n')^k]<f(x_n^k)$,即$x_n^k+\frac{1}{x_n^k}>(x_n')^k+\frac{1}{(x_n')^k}$,

$\therefore\prod_{i=1}^{n}\left(x_i^k+\frac{1}{x_i^k}\right)$

$>\left[\frac{1}{(x_n')^k}+(x_n')^k\right]\prod_{i=1}^{n-1}\left(x_i^k+\frac{1}{x_i^k}\right)\geqslant\left(n^k+\frac{1}{n^k}\right)^n$

成立,

对于5),由1),$\frac{1}{x_n'}<\frac{1}{x_n}$,$-x_n'<-x_n$,

$\therefore 0<\frac{1}{x_n'}-x_n'<\frac{1}{x_n}-x_n$,

$\therefore\prod_{i=1}^{n}\left(\frac{1}{x_i}-x_i\right)>\left(\frac{1}{x_n'}-x_n'\right)\prod_{i=1}^{n-1}\left(\frac{1}{x_i}-x_i\right)\geqslant\left(n-\frac{1}{n}\right)^k$ 成立.

例 已知:$0<x_i\leqslant 1$,$0<\sum_{i=1}^{n}\sqrt{x_i}\leqslant 1$($n\geqslant 2$且$i=1,2,\cdots,n$),

求证:$\prod_{i=1}^{n}(1-x_i)\geqslant(n^2-1)^n\prod_{i=1}^{n}x_i$.(此题摘自[2])

此不等式等价于$\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right)\geqslant(n^2-1)^n$.

令$y_i=\sqrt{x_i}$,有$\sum_{i=1}^{n}y_i\leqslant 1$,原不等式等价于

$$\prod_{i=1}^{n}\left(\frac{1}{y_i^2}-1\right)\geqslant(n^2-1)^n.$$

运用2),则水到渠成了.

参考文献:

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